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A Decomposition of the Group Algebra of a Finite Coxeter Group*

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1. INTRODUCTION

Let W be a finite Coxeter group of rank l, (see [*10*]). This means that W is a finite group with a set $R = \{r_1, ..., r_l\}$ of involutory generators and a set of defining relations of the form $(r_ir_j)^{m_{ij}} = 1$. For each subset K of $I = \{1, ..., l\}$ let W_K be the subgroup of W generated by the r_k with $k \in K$. In [*8*] I proved a character formula

$$\sum_{K \subseteq I} (-1)^{|K|} \phi_K = \epsilon, \tag{1}$$

where ϕ_K is the character of W induced by the principal character of W_K, and ϵ is the alternating character of W, the homomorphism of W into the group $\{+1, -1\}$ such that $\epsilon(r_i) = -1$ for $i \in I$. The argument avoided any close scrutiny of the characters of W and depended on application of the Hopf trace formula to the action of W on a suitable simplicial complex, the Coxeter complex of W (see [*10*, *11*]).

The present paper provides an algebraic explanation for this formula and a generalization of it conjectured in [*8*]. It also gives some new information about the group algebra of W, in particular a decomposition of the group algebra of W into a direct sum of 2^l left ideals. We begin by assuming only that W is a finite group generated by a set $R = \{r_i : i \in I\}$ and that there exists a homomorphism $\epsilon : W \to \{+1, -1\}$ with $\epsilon(r_i) = -1$ for $i \in I$. This amounts to saying that the generating set R lies outside some subgroup of index two in W. The proof of Theorem 1 requires no additional hypothesis and hence shows that formulas of type (1) exist for a wide class of finite groups. In the

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remainder of the paper we assume that W is a Coxeter group. Throughout the paper we adhere to the following notation:

$\mathbf{Z}$; $\mathbf{Q}$; $\mathbf{R}$	integers; rational numbers; real numbers
$\mid S \mid$	cardinality of a finite set S
$J, K, \ldots, P, \underset{\sim}{Q}, \ldots$	subsets of I
$\hat{J}$	complement of J in I
$A = \mathbf{Q}[W]$	group algebra of W over $\mathbf{Q}$
$A_J = \mathbf{Q}[W_J]$	
$\xi_J = \mid W_J \mid^{-1} \sum_{w \in W_J} w$	
$\eta_J = \mid W_J \mid^{-1} \sum_{w \in W_J} \epsilon(w) w$	

Both ξ_J and η_J are idempotents in the center of A_J.

THEOREM 1. *Let W be a finite group generated by a set $R = \{r_i : i \in I\}$ and let $\epsilon : W \to \{+1, -1\}$ be a homomorphism with $\epsilon(r_i) = -1$ for $i \in I$. Then the sum*

$$B = \sum_{J \subseteq I} A \xi_J \eta_{\hat{J}}$$

is direct. If ϕ_K is the character of W afforded by the left ideal $B\xi_K$ of A, then for any subset J of I we have

$$\sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \phi_K = \epsilon \sum_{\hat{J} \subseteq K \subseteq I} (-1)^{|K-\hat{J}|} \phi_K .$$

In case J is empty, this is a formula of type (1), although we cannot assert in general that ϕ_K is induced by the principal character of W_K. In fact, some of the left ideals $B\xi_K$ may be zero, in which case the corresponding ϕ_K are zero. However, since ϕ_I is the principal character and ϵ is not the principal character, the formula of Theorem 1 will always have some content. In case W is a Coxeter group, we can prove that $B = A$ and hence

THEOREM 2. *Let W be a finite Coxeter group. Then*

$$A = \sum_{J \subseteq I} A \xi_J \eta_{\hat{J}}$$

is a direct sum. If ϕ_K is the character of W induced by the principal character of W_K and ϵ is the alternating character of W, then for any subset J of I we have

$$\sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \phi_K = \epsilon \sum_{\hat{J} \subseteq K \subseteq I} (-1)^{|K-\hat{J}|} \phi_K .$$

Steinberg [9] has also given a proof of the character formula in Theorem 2. In case J is empty, this is precisely the formula (1). The following theorem and its corollary give a refined statement concerning the direct sum decomposition of Theorem 2.

THEOREM 3. *Let W be a finite Coxeter group. Let P, Q be disjoint subsets of I and suppose $i \in I$, $i \notin P \cup Q$. Let $S = P \cup \{i\}$, $T = Q \cup \{i\}$. Then we have an A-module isomorphism*

$$A\xi_P\eta_Q \simeq A\xi_S\eta_Q + A\xi_P\eta_T, \textit{ internal direct sum.}$$

COROLLARY 3.1. *Let P, Q be disjoint subsets of I. Then we have an A-module isomorphism*

$$A\xi_P\eta_Q \simeq \sum A\xi_J\eta_K, \quad \textit{internal direct sum,}$$

where the sum is over all pairs (J, K) of complementary subsets of I such that $J \supseteq P$ and $K \supseteq Q$.

Theorem 2 insures that the sum in Corollary 3.1 is direct. The corollary thus follows at once from Theorem 3, arguing by descending induction on $|P \cup Q|$. In case P and Q are empty, Corollary 3.1 amounts to the direct sum decomposition of Theorem 2. Although, on the face of it, Theorem 3 seems the natural route to Theorem 2, I have not been able to argue it in this way.

We may reformulate some of these results in terms of induced characters. In the notation of Theorem 3 write $L = P \cup Q$ and $M = L \cup \{i\}$. Let $\psi_{P,Q}$ be the character of W_L afforded by the A_L-module $A_L\xi_P\eta_Q$. Then Theorem 3, applied with W_M in place of W, states an isomorphism

$$A_M(A_L\xi_P\eta_Q) = A_M\xi_P\eta_Q \simeq A_M\xi_S\eta_Q + A_M\xi_P\eta_T.$$

Thus, letting $\mathrm{ind}_{L\to M}$ denote the induction map from characters of W_L to characters of W_M we have

COROLLARY 3.2. $\quad \mathrm{ind}_{L\to M}\psi_{P,Q} = \psi_{S,Q} + \psi_{P,T}$.

Now for $i = 0, 1,..., l$ let W_i be the group generated by $\{r_1, ..., r_i\}$ and follow the behavior of the character induced by the principal character of $W_0 = 1$ as we move up the chain of subgroups $1 = W_0 \subset W_1 \subset \cdots \subset W_l = W$. Each character "$\psi$" of W_i induces a sum of two characters "ψ" of W_{i+1}. After l steps we arrive at a splitting of the character of the regular representation of W into 2^l characters "ψ" and this is the character formulation of the direct sum decomposition of Theorem 2.

Since we have little general information about the irreducible representations of finite Coxeter groups, it would be worthwhile to try to describe the submodules of $A\xi_J\eta_J$. The following theorem gives some information in this direction. We assume that W is indecomposable, in the sense that we cannot write $I = J \cup K$ as a disjoint union of nonempty subsets such that $r_j r_k = r_k r_j$ for $j \in J$ and $k \in K$. If W is decomposable, then $W = W_J \times W_K$ and the representation theory of W reduces to that of W_J and W_K. If W is indecomposable then the Witt representation of W as a reflection group ([*10, 11*]) is an irreducible representation. Let V be the Euclidean space of dimension l which affords this representation, and let $\bigwedge^p V, p = 0, 1,..., l$, be its pth exterior power. From [*7*] we know that $\bigwedge^p V$ is an irreducible $\mathbf{R}[W]$-module and the following theorem describes the isotypic component of type $\bigwedge^p V$ in the $\mathbf{R}[W]$-module $A \otimes \mathbf{R} = \mathbf{R}[W]$. In case W is the Weyl group of a simple Lie algebra, the representation by reflections may be written in a vector space V over $\mathbf{Q}$ and there is no need to extend the ground field from $\mathbf{Q}$ to $\mathbf{R}$.

THEOREM 4. *Let W be a finite indecomposable Coxeter group. Let $\rho : W \to \mathbf{GL}(V)$ be the Witt representation of W as a reflection group, so that V is an irreducible $\mathbf{R}[W]$-module. If J is a subset of I and $p = |\hat{J}|$, then $A\xi_J\eta_{\hat{J}} \otimes \mathbf{R}$ has a unique irreducible submodule isomorphic to $\bigwedge^p V$ and has no submodule isomorphic to $\bigwedge^q V$ for $q \neq p$.*

Since the number of subsets J with $|\hat{J}| = p$ is equal to the dimension of $\bigwedge^p V$ and hence to the multiplicity of $\bigwedge^p V$ as $\mathbf{R}[W]$-submodule of $A \otimes \mathbf{R}$, we have indeed described the isotypic component of type $\bigwedge^p V$ in $A \otimes \mathbf{R}$. We have no explicit construction for the irreducible submodule of $A_J\xi\eta_{\hat{J}} \otimes \mathbf{R}$ isomorphic to $\bigwedge^p V$; its existence is shown by a character argument.

In the final section of the paper we show, in case W is the symmetric group, that the character formula of Theorem 2 amounts to an identity of MacMahon in symmetric functions, and that MacMahon has, in a sense, anticipated our decomposition of the group algebra with the introduction of an object he calls a *zigzag diagram*. For the case of the symmetric group, the character formula of Theorem 2 is closely related to an old question in combinatorics called Simon Newcomb's problem ([*3, 5*]).

2. PROOF OF THEOREM 1

The hypothesis in this section is that W is a finite group generated by a set $R = \{r_i : i \in I\}$ and that $\epsilon : W \to \{+1, -1\}$ is a homomorphism with $\epsilon(r_i) = -1$ for $i \in I$.

LEMMA 1. *Let J, K be subsets of I. If $J \cap K$ is nonempty then $\eta_K \xi_J = 0$.*

Proof. Let $L = J \cap K$. Since $W_L \subseteq W_J \cap W_K$, we may write

$W_J = \bigcup_j W_L u_j$ as a union of left cosets mod W_L and we may write $W_K = \bigcup_k v_k W_L$ as a union of right cosets mod W_L where the u_j and v_k are coset representatives. Then

$$\xi_J = | W_J : W_L |^{-1} \xi_L \left(\sum_j u_j\right) \qquad \eta_K = | W_K : W_L |^{-1} \left(\sum_k \epsilon(v_k) v_k\right) \eta_L .$$

Now the elements ξ_L, η_L are primitive idempotents in the center of A_L and since L is nonempty, ξ_L and η_L are distinct. Thus $\eta_L \xi_L = 0$ and hence $\eta_K \xi_J = 0$.

Let $*$ be the unique $\mathbf{Q}$-linear map of A into A such that $w^* = w^{-1}$ for $w \in W$. Then $*$ is an involutory antiautomorphism of A.

LEMMA 2. *Let ξ, η be idempotents of A such that $\xi^* = \xi$ and $\eta^* = \eta$. If $a \in A$ and $a(\xi\eta)^2 = 0$, then $a\xi\eta = 0$.*

Proof. If $b \in A$, let $T(b) : A \to A$ be the $\mathbf{Q}$-linear map defined by $T(b)a = ab$, $a \in A$. The bilinear form $F : A \times A \to \mathbf{Q}$ defined by $F(a, b) =$ trace $T(a^*b)$, is symmetric and positive definite. Since $F(ab, c) = F(a, cb^*)$ for $a, b, c \in A$, the transformation $T(b)$ is self-adjoint for the form F whenever $b = b^*$. Thus $T(\xi\eta\xi)$ is self-adjoint and hence, since F is positive definite, $T(\xi\eta\xi)$ is semisimple. In particular, Ker $T(\xi\eta\xi)^2 =$ Ker $T(\xi\eta\xi)$. Thus if $a(\xi\eta)^2 = 0$, then $a(\xi\eta\xi)^2 = 0$ and hence $a\xi\eta\xi = 0$. Now, using $\xi^* = \xi$ and $\eta^* = \eta = \eta^2$ we have

$$0 = F(a\xi\eta\xi, a) = F(a\xi\eta, a\xi) = F(a\xi\eta^2, a\xi) = F(a\xi\eta, a\xi\eta).$$

Thus $a\xi\eta = 0$.

LEMMA 3. *Let K be a subset of I. Then the sum*

$$\sum_{K \subseteq J \subseteq I} A\xi_J \eta_{\bar{J}} \xi_K$$

is direct.

Proof. If $K \subseteq J$, then $W_K \subseteq W_J$, so $w\xi_J = \xi_J$ whenever $w \in W_K$ and hence $\xi_K \xi_J = \xi_J$. Suppose we have a relation

$$\sum_{K \subseteq J \subseteq I} a_J \xi_J \eta_{\bar{J}} \xi_K = 0, \qquad a_J \in A.$$

For simplicity write $b_J = a_J \xi_J \eta_{\bar{J}} \xi_K$. We prove by descending induction on $| J |$ that $b_J = 0$ for all $J \supseteq K$. If $| J | = l$, then $J = I$. Multiply the given relation on the right by ξ_I. Since, by Lemma 1, $\eta_{\bar{J}} \xi_I = 0$ unless $J = I$ and then $\eta_{\bar{J}} = 1$, we conclude that $a_I \xi_I = 0$ so $b_I = 0$. Now suppose we have shown for some integer r that $b_J = 0$ whenever $J \supseteq K$ and $|J| > r$. Choose a set $L \supseteq K$ with $|L| = r$. Multiply the relation

$$\sum_{\substack{K \subseteq J \subseteq I \\ |J| \leqslant r}} b_J = 0$$

on the right by ξ_L. Since $L \supseteq K$, $\xi_K \xi_L = \xi_L$ and thus $b_J \xi_L = a_J \xi_J \eta_{\hat{J}} \xi_L$. Now, by Lemma 1, $\eta_{\hat{J}} \xi_L = 0$ unless $\hat{J} \cap L$ is empty, that is unless $L \subseteq J$. But then $r = |L| \leqslant |J| \leqslant r$ shows $L = J$. Thus $a_J \xi_J \eta_{\hat{J}} \xi_J = 0$ and *a fortiori* $a_J(\xi_J \eta_{\hat{J}})^2 = 0$. Now Lemma 2 shows $a_J \xi_J \eta_{\hat{J}} = 0$ and hence $b_J = 0$.

Let

$$B = \sum_{J \subseteq I} A \xi_J \eta_{\hat{J}}.$$

Lemma 3, applied with K empty, says that this sum is direct. This proves the first assertion of Theorem 1.

LEMMA 4. *Let K be a subset of I. Then*

$$B\xi_K = \sum_{K \subseteq J \subseteq I} A \xi_J \eta_{\hat{J}} \xi_K, \quad \textit{direct sum.}$$

Proof. This is immediate from the definition of B and Lemmas 1 and 3.

If J is a subset of I, let ϕ_J be the character of W afforded by the left ideal $B\xi_J$ and let ψ_J be the character afforded by the left ideal $A\xi_J \eta_{\hat{J}}$.

LEMMA 5. *Let K be a subset of I. Then*

$$\phi_K = \sum_{K \subseteq J \subseteq I} \psi_J$$

Proof. Suppose $J \supseteq K$. Consider the A-module epimorphism $A\xi_J \eta_{\hat{J}} \to A\xi_J \eta_{\hat{J}} \xi_K$ defined by right multiplication by ξ_K. If $a\xi_J \eta_{\hat{J}} \xi_K = 0$ for some $a \in A$ then, since $\xi_K \xi_J = \xi_J$, right multiplication by ξ_J gives $a\xi_J \eta_{\hat{J}} \xi_J = 0$, so $a(\xi_J \eta_{\hat{J}})^2 = 0$ and by Lemma 2, $a\xi_J \eta_{\hat{J}} = 0$. Thus the epimorphism is an isomorphism so that $A\xi_J \eta_{\hat{J}} \xi_K$ and $A\xi_J \eta_{\hat{J}}$ are isomorphic A-modules. The lemma is now a consequence of Lemma 4.

LEMMA 6. *Let $\theta : W \to \{+1, -1\}$ be a homomorphism and let M be an A-module which affords the character θ. Let $'$ be the unique $\mathbf{Q}$-linear map of A into A such that $w' = \theta(w)\, w^{-1}$ for $w \in W$. If $x \in A$ then Ax' and $M \otimes Ax$ are isomorphic A-modules.*

Proof. First note that the left ideal Ax and the right ideal xA afford the same character of W. One can see this by reducing to the case in which x lies in a minimal two-sided ideal of A, in which case equality of the characters amounts to equality of the dimensions of Ax and xA. Second, note that the map $'$ is an involutory antiautomorphism of A. Thus if $b_1, \ldots, b_r$ are a $\mathbf{Q}$-basis for xA, then $b_1', \ldots, b_r'$ are a $\mathbf{Q}$-basis for Ax'. Furthermore, for $w \in W$,

$$b_j w = \sum_{k=1}^{r} c_{jk}(w) b_k \Leftrightarrow \theta(w) w^{-1} b_j' = \sum_{k=1}^{r} c_{jk}(w) b_k'$$

where $c_{jk}(w) \in \mathbf{Q}$ is a matrix coefficient. Thus if Ax affords the character ψ and Ax' affords the character ψ', then

$$\psi(w) = \sum_{k=1}^{r} c_{kk}(w) \qquad \psi'(w^{-1}) = \theta(w^{-1}) \sum_{k=1}^{r} c_{kk}(w).$$

But $\theta(w^{-1}) = \theta(w)$ and since ψ' is the character of a representation of W in the rational field, we also have $\psi'(w^{-1}) = \psi'(w)$. Thus $\psi' = \theta\psi$. The lemma follows, since two A-modules are isomorphic if and only if they afford the same character of W.

Lemma 7. *If J is a subset of I, then*

$$\psi_{\bar{J}} = \epsilon \psi_J .$$

Proof. Apply Lemma 6 with $\theta = \epsilon$. If $'$ is the corresponding antiautomorphism of A, then $\xi_J' = \eta_J$ and $\eta_J' = \xi_J$. Thus $(\xi_J \eta_{\bar{J}})' = \xi_{\bar{J}} \eta_J$, whence the result.

Now we may complete the proof of Theorem 1. Let I be a finite set and let f be a function which has for domain the set of all subsets J of I and which takes values in some additive Abelian group. If

$$g(K) = \sum_{K \subseteq J \subseteq I} f(J),$$

then

$$f(J) = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} g(K).$$

This inversion formula, which may be verified directly, is a special case of a general combinatorial principle which also includes the Möbius inversion formula of elementary number theory. In our present situation the Abelian group is the additive group of generalized characters of W and we conclude from Lemma 5 that

$$\psi_J = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \phi_K .$$

Theorem 1 is now a consequence of Lemma 7.

3. Proof of Theorem 2

Let W be a finite Coxeter group with $R = \{r_i : i \in I\}$ as its set of distinguished generators. One knows ([10, 11]) that there exists a Euclidean space V of dimension l, a basis $\Pi = \{\alpha_1, ..., \alpha_l\}$ for V, and a faithful representation ρ of W by orthogonal transformations of V such that

(i) $\rho(r_i)$ is the reflection in the hyperplane orthogonal to α_i.

(ii) If $\Delta = W\Pi$ and $\alpha = c_1\alpha_1 + \cdots + c_l\alpha_l$ is the unique expression of a vector $\alpha \in \Delta$ as a linear combination of the α_i, then either all c_i are nonnegative, in which case we say $\alpha \in \Delta^+$, or all c_i are nonpositive, in which case we say $\alpha \in \Delta^-$.

The set Δ is called a *root system* for W. The vectors in Δ are called *roots* and the vectors in Π are called *simple roots*. The facts we need about root systems are summarized in the following paragraph. Proofs and further references are given in [2, 8, 11]. For simplicity of notation we write w in place of $\rho(w)$.

If $w \in W$, let $N(w)$ be the set of roots $\alpha \in \Delta^+$ such that $w\alpha \in \Delta^-$. If $N(w)$ is empty, then $w = 1$, and for $w \neq 1$ the cardinality $n(w)$ of $N(w)$ is equal to the least integer p such that $w = r_{i_1} \cdots r_{i_p}$ with $i_1, ..., i_p \in I$. If J is a subset of I, let

$$\Pi_J = \{\alpha_j : j \in J\} \qquad \Delta_J = W_J\Pi_J \qquad \Delta_J{}^+ = \Delta_J \cap \Delta^+.$$

Let V_J be the subspace of V spanned by Π_J. Then $\Delta_J = \Delta \cap V_J$. The map $w \to \rho(w) \mid V_J$, $w \in W_J$, is a representation of W_J by orthogonal transformations of V_J having properties (i) and (ii) with $W, V, \Pi, \Delta, \Delta^+$ replaced by $W_J, V_J, \Pi_J, \Delta_J, \Delta_J{}^+$. Thus theorems known to be true for W and its root system Δ may be applied to W_J and its root system Δ_J. If $w \in W_J$, then $N(w) \subseteq \Delta_J$. If J is nonempty, the group W_J has a unique involution, denoted w_J such that $N(w_J) = \Delta_J{}^+$. If $w \in W_J$ and $w \neq w_J$, then $n(w) < n(w_J)$. In [8] we introduced for each subset J of I a pair of subsets X_J, Y_J of W defined as follows: X_J consists of all $w \in W$ such that $w\Pi_J \subseteq \Delta^+$; Y_J consists of all $w \in W$ such that $w\Pi_J \subseteq \Delta^+$ and $w\Pi_{J'} \subseteq \Delta^-$. Thus, directly from the definition,

$$X_K = \bigcup_{K \subseteq J \subseteq I} Y_J, \quad \text{disjoint union.}$$

Lemma 8. *The set X_J is a set of right coset representatives for W mod W_J. If $w \in W$ and $w = xu$ with $x \in X_J$ and $u \in W_J$, then $n(w) = n(x) + n(u)$.*

Proof. See Lemma 4 of [8] and the discussion immediately following that lemma.

LEMMA 9. *Let J be a subset of I and let $K = \hat{J}$. If $y \in Y_J$, then $yw_K \in X_K$ and $n(y) = n(yw_K) + n(w_K)$.*

Proof. If $\alpha \in \Pi_K$, then $w_K\alpha \in \Delta_K^-$ so

$$w_K\alpha = \sum_{k \in K} c_k \alpha_k$$

for suitable real numbers $c_k \leqslant 0$. Then

$$yw_K\alpha = \sum_{k \in K} c_k y\alpha_k \in \Delta^+,$$

since by definition of Y_K, $y\alpha_k \in \Delta^-$. Thus $yw_K\Pi_K \subseteq \Delta^+$, so $yw_K \in X_K$. The equality $n(y) = n(yw_K) + n(w_K)$ follows now from $y = (yw_K)\, w_K$ and Lemma 8.

LEMMA 10. *Let J be a subset of I and let $K = \hat{J}$. Suppose given for each $y \in Y_J$ a rational number $c(y)$ and let*

$$\zeta_p = \sum_{y \in Y_J, n(yw_K) \geqslant p} c(y)\, y \xi_J \eta_K, \qquad p = 0, 1, 2, \ldots.$$

If $\zeta_p = 0$, then $c(y) = 0$ whenever $n(yw_K) = p$ and hence $\zeta_{p+1} = 0$.

Proof. Ignore for a moment the hypothesis $\zeta_p = 0$ and write

$$\zeta_p = \sum_{w \in W} d_p(w) w$$

for uniquely determined $d_p(w) \in \mathbf{Q}$. Suppose $z \in Y_J$ and $n(zw_K) = p$. The coefficient $d_p(zw_K)$ is equal to $|\, W_J\,|^{-1}|\, W_K\,|^{-1} \sum c(y)\, \epsilon(v)$ where the sum is over all triples (y, u, v) such that $y \in Y_J$, $u \in W_J$, $v \in W_K$, $n(yw_K) \geqslant p$ and $yuv = zw_K$. The last condition may be rewritten as $yu = zw_K v^{-1}$. Since $y \in Y_J$, $u \in W_J$ and $Y_J \subseteq X_J$, Lemma 8 implies $n(yu) = n(y) + n(u)$. Lemma 9 states $zw_K \in X_K$, so since $v^{-1} \in W_K$, we conclude again from Lemma 8 that $n(zw_K v^{-1}) = n(zw_K) + n(v^{-1})$. Lemma 9 implies $n(y) = n(yw_K) + n(w_K) \geqslant p + n(w_K)$ and, on the other hand, since $v^{-1} \in W_K$, we have $n(v^{-1}) \leqslant n(w_K)$. Thus, putting all our information together we have

$$p + n(w_K) + n(u) \leqslant n(y) + n(u) = n(zw_K) + n(v^{-1}) \leqslant p + n(w_K).$$

This forces $n(u) = 0$ so that $u = 1$, and it also forces $n(v^{-1}) = n(w_K)$ so that $v^{-1} = w_K$. Thus $v = w_K$ and $y = z$. Thus the sum over all triples consists of a single term and hence $d_p(zw_K) = |\, W_J\,|^{-1}|\, W_K\,|^{-1}\, c(z)\, \epsilon(w_K)$. Now assuming $\zeta_p = 0$, we have $d_p(w) = 0$ for all $w \in W$ so that $c(z) = 0$ whenever $n(zw_K) = p$ and the lemma is proved.

LEMMA 11. *Let J be a subset of I. The elements $y\xi_J\eta_J$, $y \in Y_J$, are a* $\mathbf{Q}$*-basis for $A\xi_J\eta_J$ and hence* $\dim A\xi_J\eta_J = |Y_J|$.

Proof. Suppose we have a relation

$$\sum_{y \in Y_J} c(y)\, y\xi_J\eta_J = 0$$

for some $c(y) \in \mathbf{Q}$. Define the ζ_p as in Lemma 10. By hypothesis $\zeta_0 = 0$, and the vanishing of all $c(y)$ is now an immediate consequence of an induction in which Lemma 10 provides the inductive step. Thus the elements $y\xi_J\eta_J$, $y \in Y_J$, are linearly independent over $\mathbf{Q}$. In particular, $\dim A\xi_J\eta_J \geqslant |Y_J|$. From Theorem 1 we know that the sum

$$B = \sum_{J \subseteq I} A\xi_J\eta_J$$

is direct so that

$$\dim A \geqslant \dim B = \sum_{J \subseteq I} \dim A\xi_J\eta_J \geqslant \sum_{J \subseteq I} |Y_J|.$$

But we see at once from the definition of the Y_J that W is a disjoint union of the Y_J, $J \subseteq I$, and hence

$$\sum_{J \subseteq I} |Y_J| = |W| = \dim A.$$

Thus we must have equality throughout, and hence $\dim A\xi_J\eta_J = |Y_J|$ so the $y\xi_J\eta_J$ are indeed a basis.

As a corollary of the preceding argument we also have

$$A = B = \sum_{J \subseteq I} A\xi_J\eta_J.$$

This proves the first assertion in Theorem 2. Since ϕ_K is thus the character of W afforded by the left ideal $A\xi_K$, ϕ_K may also be interpreted as the character of W induced by the principal character of W_K and this completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

LEMMA 12. *If J, K are subsets of I, then $A\xi_J\eta_K$ and $A\eta_K\xi_J$ are isomorphic A-modules.*

Proof. Apply Lemma 6 with $\theta(w) = 1$ for all $w \in W$. If $'$ is the corresponding antiautomorphism of A, then $x' = x^*$ for $x \in A$. Since $(\xi_J\eta_K)^* = \eta_K\xi_J$, we have the result.

Theorem 3 is now a consequence of the following statements numbered (i)–(iv), in which we adhere to the notation of the theorem:

(i) The sums $A\xi_S\eta_Q + A\xi_P\eta_T$ and $A\xi_S\eta_Q + A\eta_T\xi_P\eta_Q$ are direct: For if $a\xi_S\eta_Q = b\xi_P\eta_T$ with $a, b \in A$, then since $T \supseteq Q$, we have $\eta_Q\eta_T = \eta_T$ and right multiplication by η_T gives $a\xi_S\eta_T = b\xi_P\eta_T$. But $S \cap T$ is nonempty, so Lemma 1 shows $a\xi_S\eta_T = 0$ whence $b\xi_P\eta_T = 0$ and the sum $A\xi_S\eta_Q + A\xi_P\eta_T$ is direct. Similarly, if $a\xi_S\eta_Q = b\eta_T\xi_P\eta_Q$ with $a, b \in A$, then right multiplication by η_T shows $b\eta_T\xi_P\eta_T = 0$. Then $b(\eta_T\xi_P)^2 = 0$ and by Lemma 2 (with ξ and η interchanged) we have $b\eta_T\xi_P = 0$ so the sum $A\xi_S\eta_Q + A\eta_T\xi_P\eta_Q$ is direct.

(ii) $A\eta_T\xi_P\eta_Q$ and $A\xi_P\eta_T$ are isomorphic A-modules: Right multiplication by η_T defines A-module epimorphisms $A\eta_T\xi_P\eta_Q \to A\eta_T\xi_P\eta_T$ and $A\eta_T\xi_P \to A\eta_T\xi_P\eta_T$. Using Lemma 3 we see, as in the proof of Lemma 5, that these epimorphisms are isomorphisms. Then Lemma 12 gives $A\xi_P\eta_T \simeq A\eta_T\xi_P \simeq A\eta_T\xi_P\eta_Q$.

(iii) $A\xi_P\eta_Q \supseteq A\xi_S\eta_Q + A\eta_T\xi_P\eta_Q$: This is clear since $S \supseteq P$ and hence $A\xi_S \subseteq A\xi_P$.

(iv) $\dim A\xi_P\eta_Q = \dim A\xi_S\eta_Q + \dim A\xi_P\eta_T$: We introduce some notation. If J, L are subsets of I with $J \subseteq L$, let

$$X_{L,J} = \{w \in W_L : w\Pi_J \subseteq \Delta^+\}.$$

If J, K are disjoint subsets of I, let

$$Y_{J,K} = \{w \in W_{J\cup K} : w\Pi_J \subseteq \Delta^+ \text{ and } w\Pi_K \subseteq \Delta^-\}.$$

Thus if $L = I = J \cup K$, we have $X_{L,J} = X_J$ and $Y_{J,K} = Y_J$ in our earlier notation. Now in the notation of Theorem 3, let $L = P \cup Q$ and let $M = L \cup \{i\}$. Suppose $x \in X_{M,L}$ and $y \in Y_{P,Q}$. Then by definition $y\alpha_j \in \Delta^+$ for $j \in P$, $y\alpha_j \in \Delta^-$ for $j \in Q$, and $x\alpha_j \in \Delta^+$ for $j \in P \cup Q$. Since $y \in W_L$ we have $y\Delta_L \subseteq \Delta_L$, and since x preserves the sign of all roots in Δ_L we have $xy\alpha_j \in \Delta^+$ for $j \in P$ and $xy\alpha_j \in \Delta^-$ for $j \in Q$. Since $xy \in W_M$ it follows that $xy \in Y_{S,Q}$ if $xy\alpha_i \in \Delta^+$, and $xy \in Y_{P,T}$ if $xy\alpha_i \in \Delta^-$. Thus

$$X_{M,L}Y_{P,Q} \subseteq Y_{S,Q} \cup Y_{P,T}.$$

Since $Y_{P,Q} \subseteq W_L$ we conclude from Lemma 8 applied to the group W_M and subgroup W_L that

$$| X_{M,L} | = | W_M : W_L | \text{ and } | X_{M,L}Y_{P,Q} | = | X_{M,L} | \, | Y_{P,Q} |$$

so that

$$| W_M : W_L | \, | Y_{P,Q} | \leqslant | Y_{S,Q} | + | Y_{P,T} |.$$

Now hold L fixed, let (P, Q) vary over all pairs of complementary subsets of L and let us agree that for any given (P, Q) we set $S = P \cup \{i\}$, $T = Q \cup \{i\}$, and $M = L \cup \{i\}$. Then summing the preceding inequality over all such pairs (P, Q) gives the inequality $| W_M : W_L | \, | W_L | \leqslant | W_M |$. But this is actually an equality, so we must have equality throughout and hence

$$| W_M : W_L | \, | Y_{P,Q} | = | Y_{S,Q} | + | Y_{P,T} |.$$

Then using Lemma 11 for the groups W_L and W_M in place of W we have

$$\begin{aligned}\dim A\xi_P\eta_Q &= | W : W_L | \dim A_L\xi_P\eta_Q \\ &= | W : W_L | \, | Y_{P,Q} | \\ &= | W : W_M | \, | Y_{S,Q} | + | W : W_M | \, | Y_{P,T} | \\ &= | W : W_M | \dim A_M\xi_S\eta_Q + | W : W_M | \dim A_M\xi_P\eta_T \\ &= \dim A\xi_S\eta_Q + \dim A\xi_P\eta_T .\end{aligned}$$

This completes the proof of Theorem 3.

5. Proof of Theorem 4

Let W be a finite indecomposable Coxeter group. We adhere to the notation of the previous section so that V is a Euclidean space in which the r_i, $i \in I$, act as reflections, and we identify W with the corresponding group of orthogonal transformations of V. Since W is indecomposable, V is an irreducible $\mathbf{R}[W]$-module ([*11*]). Let $(\cdot\ ,\ \cdot)$ denote the inner product in V and let $\beta_1 ,..., \beta_l$ be the basis for V dual to $\alpha_1 ,..., \alpha_l$ so that $(\alpha_i , \beta_j) = \delta_{ij}$. The cone spanned by $\beta_1 ,..., \beta_l$ is a fundamental region for W in its action on V (see [*11*]). If K is a subset of I, then the group W_K generated by the r_k with $k \in K$ may also be described as the subgroup of W fixing the β_k with $k \in \hat{K}$ (see [*11*]).

In this section W_K-module means $\mathbf{R}[W_K]$-module. If M is a W_K-module, we let $\mathbf{I}_K(M)$ denote the subspace of W_K-invariants, elements $x \in M$ such that $wx = x$ for all $w \in W_K$. If $M = M_1 + M_2$ is a direct sum of W_K-modules, then $\mathbf{I}_K(M) = \mathbf{I}_K(M_1) + \mathbf{I}_K(M_2)$.

Lemma 13. *If $J \subseteq K \subseteq I$, then* $\dim I_J(V_K) = | K | - | J |$.

Proof. Since r_i is the reflection in the hyperplane orthogonal to α_i we have

$$r_i v = v - 2(\alpha_i , \alpha_i)^{-1}(v, \alpha_i)\, \alpha_i , \qquad v \in V.$$

Thus r_i fixes $v \in V$ if and only if $(v, \alpha_i) = 0$ and hence $\dim \mathbf{I}_J(V) = l - |J|$. Now $\{\alpha_K : k \in K\} \cup \{\beta_k : k \in \hat{K}\}$ is a basis for V so that

$$V = V_K + \sum_{k \in \hat{K}} \mathbf{R}\beta_k,$$

which is a direct sum of vector spaces. But W_K and *a fortiori* W_J fixes

$$\sum_{k \in \hat{K}} \mathbf{R}\beta_k.$$

Since V_K is a W_J-module this means that the direct sum is one of W_J-modules. Hence

$$\mathbf{I}_J(V) = \mathbf{I}_J(V_K) + \mathbf{I}_J\Big(\sum_{k \in \hat{K}} \mathbf{R}\beta_k\Big).$$

Equating dimensions gives

$$l - |J| = \dim \mathbf{I}_J(V_K) + (l - |K|)$$

and the lemma is proved.

Lemma 14. *Let K be a nonempty subset of I. Then $\mathbf{I}_K(\bigwedge^p V_K) = 0$ for all $p = 1, 2, \ldots$.*

Proof. Argue by induction on $|K|$, and for fixed $|K|$ by induction on p. For $|K| = 1$, W_K is generated by a single r_k and V_K is spanned by α_k, so since $r_k\alpha_k = -\alpha_k$, the result is clear. For $|K| \geqslant 2$ and $p = 1$ the statement is a consequence of Lemma 13. Thus we may assume $|K| \geqslant 2$ and $p \geqslant 2$. Let J be a subset of K with $|J| = |K| - 1$. Then Lemma 13 shows $\dim \mathbf{I}_J(V_K) = 1$ so we may choose a vector $v \in V_K$ which is fixed by W_J. Since Lemma 13 shows $\mathbf{I}_J(V_J) = 0$, it follows that $v \notin V_J$ so $V_K = V_J + \mathbf{R}v$ and thus

$$\textstyle\bigwedge^p V_K = \bigwedge^p V_J + (\bigwedge^{p-1} V_J \wedge v)$$

where the sums are direct. Since W_J fixes v, both summands are W_J-modules so

$$\begin{aligned}\mathbf{I}_J(\textstyle\bigwedge^p V_K) &= \mathbf{I}_J(\textstyle\bigwedge^p V_J) + \mathbf{I}_J(\textstyle\bigwedge^{p-1} V_J \wedge v)\\ &= \mathbf{I}_J(\textstyle\bigwedge^p V_J) + (\mathbf{I}_J(\textstyle\bigwedge^{p-1} V_J) \wedge v).\end{aligned}$$

Our induction hypothesis implies $\mathbf{I}_J(\bigwedge^p V_J) = 0$ and since $p \geqslant 2$ it also implies $\mathbf{I}_J(\bigwedge^{p-1} V_J) = 0$. Thus $\mathbf{I}_J(\bigwedge^p V_K) = 0$ and *a fortiori* $\mathbf{I}_K(\bigwedge^p V_K) = 0$.

If $K = \{k_1, \ldots, k_p\}$ is a subset of I and $k_1 < \cdots < k_p$, write

$$\alpha_K = \alpha_{k_1} \wedge \cdots \wedge \alpha_{k_p} \qquad \text{and} \qquad \beta_K = \beta_{k_1} \wedge \cdots \wedge \beta_{k_p}.$$

If K is empty we agree that $\alpha_K = 1 = \beta_K$, identifying $\bigwedge^0 V$ with $\mathbf{R}$.

LEMMA 15. *Let K be a subset of I. Then* $\mathbf{I}_K(\bigwedge^p V) = 0$ *for* $|\hat{K}| < p$, *while* $\mathbf{I}_K(\bigwedge^p V) = \mathbf{R}\beta_{\hat{K}}$ *for* $|\hat{K}| = p$.

Proof. Since $\{\alpha_k : k \in K\} \cup \{\beta_k : k \in \hat{K}\}$ is a basis for V it follows that the elements $\alpha_M \wedge \beta_N$ with $M \subseteq K$, $N \subseteq \hat{K}$ and $|M| + |N| = p$ are a basis for $\bigwedge^p V$. For each $N \subseteq \hat{K}$ with $|N| \leqslant p$ let

$$U_N = \sum_M \mathbf{R}(\alpha_M \wedge \beta_N)$$

where the direct sum is over all $M \subseteq K$ with $|M| + |N| = p$. Thus $\bigwedge^p V = \sum_N U_N$ where the direct sum is over all $N \subseteq \hat{K}$ with $|N| \leqslant p$. Since W_K fixes all β_k with $k \in \hat{K}$, it fixes all β_N with $N \subseteq \hat{K}$. Furthermore, since V_K is a W_K-module, $\sum_M \mathbf{R}\alpha_M$ is a W_K-module and hence so is U_N. Thus

$$\mathbf{I}_K(\textstyle\bigwedge^p V) = \displaystyle\sum_N \mathbf{I}_K(U_N).$$

Let $\gamma \in \mathbf{I}_K(\bigwedge^p V)$ and write $\gamma = \sum_N \gamma_N$ where $\gamma_N \in \mathbf{I}_K(U_N)$. Write $\gamma_N = \sum_M c_{M,N}\alpha_M \wedge \beta_N$ where $c_{M,N} \in \mathbf{R}$ and the sum is over all $M \subseteq K$ with $|M| + |N| = p$. Then setting $\delta_N = \sum_M c_{M,N}\gamma_M$ we have $\delta_N \in \bigwedge^q V_K$ where $q = p - |N|$. In case N is empty we agree that $\delta_N = 1$.

If $w \in W_K$, then $w\beta_N = \beta_N$ so $(w\delta_N - \delta_N) \wedge \beta_N = w\gamma_N - \gamma_N = 0$. But V_K is a W_K-module so $w\delta_N - \delta_N$ is a linear combination of the α_M with $M \subseteq K$, and since the $\alpha_M \wedge \beta_N$ are linearly independent we have $w\delta_N - \delta_N = 0$. Thus $\delta_N \in \mathbf{I}_K(\bigwedge^q V_K)$. It follows from Lemma 14 that δ_N and hence γ_N is zero whenever $q = p - |N|$ is positive.

If $|\hat{K}| < p$, then $|N| < p$ for all $N \subseteq \hat{K}$ so $\gamma_N = 0$ for all $N \subseteq \hat{K}$ and hence $\gamma = 0$. Thus $\mathbf{I}_K(\bigwedge^p V) = 0$ for $|\hat{K}| < p$. If $|\hat{K}| = p$, then $|N| < p$ for all $N \subseteq \hat{K}$ except $N = \hat{K}$ and hence $\gamma \in \mathbf{R}\beta_{\hat{K}}$. Since $\beta_{\hat{K}} \in \mathbf{I}_K(\bigwedge^p V)$ we have $\mathbf{I}_K(\bigwedge^p V) = \mathbf{R}\beta_{\hat{K}}$ and the lemma is proved.

Now to prove Theorem 4, let χ be the character of W afforded by V and let $\bigwedge^p \chi$ be the character of W afforded by $\bigwedge^p V$. We have shown in the proof of Theorem 2 that

$$\psi_J = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|}\phi_K.$$

Thus letting $(\bigwedge^p \chi, \psi_J)$ denote the multiplicity of $\bigwedge^p \chi$ as irreducible constituent of ψ_J we have

$$(\textstyle\bigwedge^p \chi, \psi_J) = \displaystyle\sum_{J \subseteq K \subseteq I} (-1)^{|K-J|}(\textstyle\bigwedge^p \chi, \phi_K).$$

The multiplicity $(\bigwedge^p \chi, \phi_K)$ is, by the Frobenius reciprocity theorem, equal to the multiplicity of the principal character of W_K in the restriction of $\bigwedge^p \chi$

to W_K, and hence equal to $\dim \mathbf{I}_K(\bigwedge^p V)$. Thus

$$(\textstyle\bigwedge^p \chi, \psi_J) = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \dim \mathbf{I}_K(\textstyle\bigwedge^p V).$$

If $|\hat{J}| = p$, then Lemma 15 shows $\dim \mathbf{I}_J(\bigwedge^p V) = 1$, while $\dim \mathbf{I}_K(\bigwedge^p V) = 0$ whenever $J \subset K$. Thus if $|\hat{J}| = p$, then $(\bigwedge^p \chi, \psi_J) = 1$, so that the W_J-module $A\xi_J\eta_J \otimes \mathbf{R}$, which affords the character ψ_J, has a unique submodule, call it M_J, isomorphic to $\bigwedge^p V$. But $A \otimes \mathbf{R}$ is a module for the regular representation of W so the isotypic component of $A \otimes \mathbf{R}$ of type $\bigwedge^p V$ is isomorphic to a direct sum of $\dim \bigwedge^p V = \binom{l}{p}$ copies of $\bigwedge^p V$. Since Theorem 2 insures that the sum

$$\sum_{|\hat{J}|=p} M_J$$

is direct; it is in fact the isotypic component of type $\bigwedge^p V$. Thus if $q \neq p$, $A\xi_J\eta_{\hat{J}}$ can have no submodule isomorphic to $\bigwedge^q V$. This completes the proof of Theorem 4.

6. The Symmetric Group

The source of Simon Newcomb's problem ([4], §156) is a game of patience which Professor Newcomb would play as a respite from his astronomical work. A deck of cards marked $1, \ldots, n$ is shuffled[1] and the cards are dealt one by one. As long as the cards occur in increasing order, one puts them into a single pile. As soon as there is a break in the increasing order, one starts a new pile, deals until there is a break in the increasing order, starts a new pile, and so on. The problem is to determine for any integer $k = 1, \ldots, n$ the probability that there are precisely k piles.

Let $W = \mathfrak{S}_n$ be the symmetric group on the letters $1, \ldots, n$ and let $r_i \in W$ be the transposition of the letters i and $i + 1$. Then W is an indecomposable Coxeter group of rank $l = n - 1$ on the generators r_i. The irreducible representation of W as a reflection group may be described as follows. Let $\lambda_1, \ldots, \lambda_n$ be an orthonormal basis for a Euclidean space E of dimension n. Give E a W-module structure by defining $w\lambda_i = \lambda_{w(i)}$ where $w(i)$ denotes the image of the letter i under the permutation w. Let V be the subspace of dimension $n - 1$ spanned by the vectors $\lambda_i - \lambda_j$, $i, j = 1, \ldots, n$. Then V is an irreducible W-module and the r_i, restricted to V, are reflections. One may take $I = \{1, \ldots, n - 1\}$, $\alpha_i = \lambda_i - \lambda_{i+1}$, $i \in I$, and then

$$\Delta^+ = \{\lambda_i - \lambda_j : i < j\} \qquad \Delta^- = \{\lambda_i - \lambda_j : i > j\}$$

[1] One might use $l + 1$ instead of n here. This would give us a notation more in keeping with the earlier sections but less natural for the symmetric group.

In Simon Newcomb's problem, each $w \in W$ defines an initial ordering $w(1), \ldots, w(n)$ of the deck, and hence an ordered collection $\mathbf{p} = (p_1, \ldots, p_k)$ of positive integers, where p_i is the number of cards in the ith pile and $k = k(w)$ is the number of piles. Since every card occurs in some pile we have $p_1 + \cdots + p_k = n$. From the definition of the p_i we have inequalities

$$\begin{array}{ll} w(1) < \cdots < w(p_1) & w(p_1) > w(p_1 + 1) \\ w(p_1 + 1) < \cdots < w(p_1 + p_2) & w(p_1 + p_2) > w(p_1 + p_2 + 1) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ w(p_1 + p_2 + \cdots + p_{k-1} + 1) < \cdots < w(p_1 + \cdots + p_k) = w(n) & \end{array}$$

so that the elements of Π which are mapped into Δ^- by w are

$$\alpha_{p_1}, \alpha_{p_1+p_2}, \ldots, \alpha_{p_1+p_2+\cdots+p_{k-1}}.$$

Setting $J = \{p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_{k-1}\}$ we see that w defines the ordered collection $\mathbf{p} = (p_1, \ldots, p_k)$ if and only if $w \in Y_{\bar{J}}$. If w_I is the involution in W which changes the sign of all the roots, then $Y_{\bar{J}} = w_I Y_J$ so $| Y_{\bar{J}} | = | Y_J |$. Thus for a given integer $k \in \{1, \ldots, n\}$ the number of elements $w \in W$ which produce k piles is

$$\sum_{|J|=k-1} | Y_J |.$$

Simon Newcomb's problem amounts to computing this number and hence to computing the numbers $| Y_J |$. Since

$$| X_K | = \sum_{K \subseteq J \subseteq I} | Y_J |,$$

we conclude, using a Möbius inversion and Lemma 8, that

$$| Y_J | = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} | X_K | = \sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} | W : W_K |.$$

The $| Y_J |$ are thus computable in terms of the known orders of certain subgroups of the symmetric group. MacMahon's solution ([*4*], §157) is purely combinatorial in the sense that he does not use any of the group structure, but his argument also involves a Möbius inversion and is substantially the same as the one above.

To see the connection between Theorem 2 and MacMahon's work we translate the character formula of Theorem 2 into the language of symmetric functions. The next two paragraphs describe the necessary apparatus of Schur functions; proofs are in Littlewood's work, [*3*].

Let $\mathfrak{S}_m$, $m = 1, 2, 3,\ldots$ be the symmetric group on m letters. Let $\mathfrak{X}_m$ be the free $\mathbf{Z}$-module which has as its basis the set of irreducible characters of $\mathfrak{S}_m$ and let us agree that $\mathfrak{X}_0 = \mathbf{Z}$. We agree to ignore the natural multiplicative structure in $\mathfrak{X}_m$. If ω is a character of $\mathfrak{S}_m$ and ζ is a character of $\mathfrak{S}_n$, let $\omega \times \zeta$ denote the character of $\mathfrak{S}_m \times \mathfrak{S}_n$ defined by $(\omega \times \zeta)(wz) = \omega(w)\,\zeta(z)$ for $w \in \mathfrak{S}_m$ and $z \in \mathfrak{S}_n$. We may view $\mathfrak{S}_m \times \mathfrak{S}_n$ as a subgroup of $\mathfrak{S}_{m+n}$ by letting $\mathfrak{S}_m$ act on the letters $1,\ldots, m$, and $\mathfrak{S}_n$ on the letters $m+1,\ldots, m+n$. Then $\omega \times \zeta$ induces a character of $\mathfrak{S}_{m+n}$ which we denote $\omega\zeta$, and $(\omega, \zeta) \to \omega\zeta$ defines a $\mathbf{Z}$-bilinear map of $\mathfrak{X}_m \times \mathfrak{X}_n$ into $\mathfrak{X}_{m+n}$. With this bilinear map as multiplication, the graded $\mathbf{Z}$-module

$$\mathfrak{X} = \sum_{m \geqslant 0} \mathfrak{X}_m$$

becomes an associative commutative graded $\mathbf{Z}$-algebra called the *Schur algebra*.

Now choose a fixed integer n and let $x_1,\ldots, x_n$ be commuting indeterminates over $\mathbf{Z}$. The group $\mathfrak{S}_n$ acts naturally as a group of automorphisms of the polynomial algebra $\mathbf{Z}[x_1,\ldots, x_n]$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{S}_n$-invariant polynomials, the so-called symmetric functions of the indeterminates $x_1,\ldots, x_n$. Then

$$\mathfrak{A} = \sum_{m \geqslant 0} \mathfrak{A}_m$$

where $\mathfrak{A}_m$ consists of the homogeneous invariants of degree m. Let

$$h_m = \sum_{i_1 \leqslant \cdots \leqslant i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \in \mathfrak{A}_m, \qquad m = 1, 2, 3,\ldots,$$

$$a_m = \sum_{i_1 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \in \mathfrak{A}_m, \qquad m = 1,\ldots, n.$$

We agree that $h_0 = 1 = a_0$. The graded $\mathbf{Z}$-algebra $\mathfrak{A}$ is generated by either $h_1,\ldots, h_n$ or $a_1,\ldots, a_n$, and the identity. Let

$$s_m = \sum_{i=1}^{n} x_i^m \in \mathfrak{A}_m, \qquad m = 1, 2, 3,\ldots.$$

If $w \in \mathfrak{S}_m$, let $\gamma_i(w)$, $i = 1,\ldots, m$ be the number of cycles of length i in the permutation w. For $\chi \in \mathfrak{X}_m$ let

$$S_m(\chi) = \frac{1}{m!} \sum_{w \in \mathfrak{S}_m} \chi(w) s_1^{\gamma_1(w)} s_2^{\gamma_2(w)} \cdots s_m^{\gamma_m(w)}$$

be the *Schur function* associated with the character χ. From the definition we know only that $S_m(\chi) \in \mathbf{Q}\mathfrak{A}_m \subseteq \mathbf{Q}[x_1,\ldots, x_n]$; but in fact, $S_m(\chi) \in \mathfrak{A}_m$ so that S_m is a $\mathbf{Z}$-linear map of $\mathfrak{X}_m$ into $\mathfrak{A}_m$. In case $m = 0$ we agree that S_0 maps the identity element of $\mathfrak{X}_0$ into the identity element of $\mathfrak{A}_0$. The direct sum S of the S_m is a $\mathbf{Z}$-linear map of $\mathfrak{X}$ into $\mathfrak{A}$ which is in fact an epimorphism of graded algebras, called the *Schur homomorphism*. The Schur homomorphism maps the principal character of $\mathfrak{S}_m$ into h_m and maps the alternating character of $\mathfrak{S}_m$ into a_m. It is an isomorphism in degrees 0, 1,..., n and thus allows one to translate formulas involving characters of $\mathfrak{S}_n$ into identities in the algebra $\mathfrak{A}$.

A *composition* of the integer n is an ordered collection $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ of positive integers with sum equal to n. The p_i are called the *parts* of the composition. MacMahon ([*4*], §§168–171) has defined for each composition $\mathbf{p}$ a pair of symmetric functions $h_\mathbf{p}$, $a_\mathbf{p}$ as follows: $h_\mathbf{p}$ is the determinant

$$\begin{vmatrix} h_{p_1} & h_{p_1+p_2} & h_{p_1+p_2+p_3} & h_{p_1+p_2+p_3+p_4} & \cdots \\ 1 & h_{p_2} & h_{p_2+p_3} & h_{p_2+p_3+p_4} & \cdots \\ 0 & 1 & h_{p_3} & h_{p_3+p_4} & \cdots \\ 0 & 0 & 1 & h_{p_4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

and a_p is the corresponding determinant in which the h_m are replaced by the a_m.

Each composition $\mathbf{p}$ of n determines a subset of the set I which we denote $F(p)$ and define as the complement of the set $\{p_1, p_1 + p_2, p_1 + p_2 + p_3, \ldots\}$. The correspondence $\mathbf{p} \to F(\mathbf{p})$ is suggested by the solution of Simon Newcomb's problem and is a one-to-one correspondence between the set of all compositions of n and the set of all subsets of I (see [*4*], §125). We define the composition $\hat{\mathbf{p}}$ *conjugate* to the composition $\mathbf{p}$ by requiring $F(\hat{\mathbf{p}})$ to be the complement of $F(\mathbf{p})$ in I (see [*4*], §126). MacMahon has shown ([*4*], §168) that the determinant $h_\mathbf{p}$ may also be written as

$$h_\mathbf{p} = \sum_{F(\mathbf{q}) \supseteq F(\mathbf{p})} (-1)^{|F(\mathbf{q}) - F(\mathbf{p})|} h_{q_1} h_{q_2} h_{q_3}, \ldots,$$

where the sum is over all compositions $\mathbf{q} = (q_1, q_2, q_3, \ldots)$ of n such that $F(\mathbf{q}) \supseteq F(\mathbf{p})$. One has a corresponding formula for the $a_\mathbf{p}$. Since S maps the principal character of $\mathfrak{S}_m$ into h_m and the alternating character of $\mathfrak{S}_m$ into a_m we conclude, setting $K = F(\mathbf{q})$, that

$$S(\phi_K) = h_{q_1} h_{q_2} h_{q_3} \cdots \qquad S(\epsilon\phi_K) = a_{q_1} a_{q_2} a_{q_3}, \ldots.$$

Setting $J = F(\mathbf{p})$, we have

$$h_\mathbf{p} = S\Big(\sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \phi_K\Big) = S(\psi_J)$$

$$a_\mathbf{p} = S\Big(\sum_{J \subseteq K \subseteq I} (-1)^{|K-J|} \epsilon\phi_K\Big) = S(\epsilon\psi_J).$$

Since $\psi_{\hat{J}} = \epsilon\psi_J$ and $\hat{J} = F(\hat{\mathbf{p}})$ the character formula of Theorem 2 is thus equivalent to the identity $h_{\mathbf{p}} = a_{\hat{\mathbf{p}}}$ in symmetric functions, which MacMahon proves ([*4*], §173) by induction on the number of parts in $\mathbf{p}$. Aitken [*1*] gives a direct proof of the identity as a consequence of a general theorem on determinants whose entries are the symmetric functions h_m and a_m.

As for the left ideals $A\xi_J\eta_{\hat{J}}$, MacMahon anticipates their construction in the following way (see [*4*], §129): To each composition $\mathbf{p}$ of n he associates a diagram of n nodes. There are p_i nodes in the ith row of the diagram and the right end node in any row lies immediately above the left end node in the row beneath. Thus the MacMahon diagram corresponding to the composition (4, 2, 3) is

$$\begin{array}{cccccc} \cdot & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \;\cdot \end{array}$$

The diagram of $\hat{\mathbf{p}}$ is the transpose of the diagram of $\mathbf{p}$. Now let us replace the nodes in the diagram by the integers 1,..., n in such a way that the integers increase from left to right in each row and from top to bottom in each column. This can be done in just one way. Call the resulting figure the standard MacMahon tableau and denote it $M(\mathbf{p})$. Thus

$$M(4, 2, 3) = \begin{array}{cccccc} 1 & 2 & 3 & 4 & & \\ & & & 5 & 6 & \\ & & & & 7 & 8 \;\; 9 \end{array} .$$

If $J = F(\mathbf{p})$ is the subset of I corresponding to the composition $\mathbf{p}$, then W_J is the subgroup of W leaving the rows of $M(\mathbf{p})$ invariant and $W_{\hat{J}}$ is the subgroup of W leaving the columns of $M(\mathbf{p})$ invariant. The left ideals $A\xi_J\eta_{\hat{J}}$ are thus defined by the standard MacMahon tableau in the same way that minimal left ideals of A are defined by the standard Young tableaux [*3*]. The direct decomposition of Theorem 2 may thus be viewed as an analog of Young's direct decomposition of A into minimal left ideals.

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